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Newton’s Method and Chaotic Behavior

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Chapter 1

Introduction

1.1 Abstract

This project will explore why, when using an iterative algorithm, specifically Newton's Method, to solve nonlinear equations, certain functions can be observed to behave predictably while others behave chaotically. In attempting to answer this query my project will elaborate on what Newton's Method is and how its used as well as demonstrate that Newton's Method itself behaves predictably via mathematical proof. In this context, I will examine real-valued functions with solutions then introduce complex-valued functions. Following a proof of Newton's Method for complex functions, the project will compare the behavior of these complex-valued functions with the previously mentioned real-valued functions. The project will examine the convergence behavior of Newton's Method when analyzing complex-valued functions and determine if the behavior is chaotic. Upon observing this chaotic

behavior, my project will seek to find complex functions that do not exhibit chaotic convergence behavior.

Subsequently, I will analyze my discoveries and discuss their implications. As of yet, there has been no comprehensive study of iterative methods in the context of solving complex valued equations. Ultimately, my project will produce an analytic discussion of the behavior of several functions within Newton's Method along with computational experiments. From this process we might find some distinguishing factor that determines whether behavior will be predictable or chaotic.

1.2 Overview

The world of chaos and complex numbers appears elusive, and secondary to the real world that undergraduate students typically work within. However, this is a misconception. We see integers, rational and irrational numbers, we understand and need negative numbers so, as a whole, we understand all of the real numbers.

After all that, we come to the need for imaginary and complex numbers. The name is deceiving because these numbers don't represent merely imaginary things that don't effect the world, they exists and they have effect. Complex numbers include within them all the other number types that we know and love. They can be represented as: $a+bi$ where a and b are real numbers, a is the real component and b is the imaginary component. So, since b can equal zero, $a+bi$ can represent all real numbers. Thus, when we're working in the world of real numbers, because complex numbers are simply an abstract extension of real numbers, it's not impossible to start with a real-valued object and find complex-valued results (as we might see in our complex-valued examples). In seeing this, it emphasizes the idea that the real and the complex are not completely separate sets. Our abstract outlines the intentions for this project but it's not a mutually-exclusive demonstration of what this project encapsulates. This work attempted to meticulously go over the use of Newton's Method and what this tool was used to analyze. We then explored our examples and found only complex-valued functions to exhibit chaotic behavior. But we were able to reveal that chaotic behavior (and its fractal structure) using a point-wise definition of Chaos. Consequently we analyzed our discoveries and discuss the implications of what we found in relation to our central question: How it is that, using the same iterative algorithm to find the roots of a complex valued function, we can see either predictable or unpredictable (chaotic) behavior?

1.3 Background Knowledge

In order to understand the proceeding work and analysis first recall complex numbers and functions. A paper called *Fractal Newton Basins* by M. L. Sahari and I. Djellit was used often as inspiration and reference for this project so it's significant to see what that paper said of complex functions and Newton's Method, how they went about their work, the paper's structure, what their results meant for them and what it means for this project. The contents of this paper will not be reviewed at length in this work but will be broadly mentioned at times. Further, it will prove critical to have an understanding of the planes in which the complex numbers exist, how to convert between rectangular and polar form of these numbers, De Moivre's Theorem and it's use, and how to find complex roots of a complex function. These will all be addressed in a logical section or defined in a clearly identifiable term box. We will also need to understand Newton's iterative algorithm versus the Secant Method. Though, our work never ran into issues with Newton's Method, and thus had no need for the Secant Method, it's important to make note of the resource being available and elaborate on how it could have been utilized. Additionally, as the work progresses away from real-valued functions, it will be necessary to have an understanding of mathematical Chaos. We will both define this in a term box but also elaborate on Chaos when we see this behavior in our explorations.

Chapter 2

How Does Newton's Method Work?

Iterative Algorithm:

A mathematical process or system which establishes a sequence of approximation, in our case, of solutions (roots) beginning with an initial step, each step using previous steps and each getting more accurate (closer to the root) until a determined sufficiently close end point is reached.

Newton's Method:

"A classical algorithm for finding roots of a function." [2]. An iterative algorithm for finding the roots of real or complex valued functions. The simple theorem we will use to define Newton's Method is as follows : Suppose F is a function. The Newtons iterations function associated to F is the function

$$N(x) = x - \frac{F(x)}{F'(x)}.$$

[Further Explanation (FE): Iterative means using iterations, which are simply the steps of our processes. Algorithm means, in our case, a system or process we work through however many times necessary to reach our answer or determine there is no answer using this process. So, iterations can be understood as the number of times we use the algorithm.]

See Proof at beginning of Real Valued Functions section for further understanding.

In the context of solving equations, Newton's Method works by starting with a given initial guess, a point, and finding the value of a function at that point (take an x value and find the y value). Next, the method finds the derivative (the slope) at that point and creates a tangent line (line that touches the graph at a point, having the slope of the function at that point). Now, the method takes the value at which this tangent line crosses the x axis; that is one iteration. Newton's Method begins the process again with this new x-value, until it finds a solution or doesn't. The algorithm ends when it arrives at a root, namely when $|x_n - \alpha| < \epsilon$ where x_n is the approximation, α is the desired root, and ϵ is the desired level of accuracy. In practice, this is often measured by $|x_n - x_{n-1}| < \epsilon$ where x_n is the newest x-value the algorithm has found, so x_{n-1} is the previous x-value, and ϵ is the stopping tolerance. It's important to notice that Newton's Method requires a non-zero derivative to converge to the root of a function, as is made clear by the equation above where the derivative is in the denominator. [FE: Besides seeing that the equation for Newton's Method has the derivative of the function as a denominator, we can understand why Newton's Method requires a non-zero derivative by visualizing what happens in this case. When the derivative at a point is zero, the tangent line at that point is horizontal. Meaning, that the tangent line won't cross the x-axis and there will be no successive x-intercept.]

Secant Method: [1]

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})} \text{ [1]}$$

[FE: Secant Method works essentially the same way as Newton's Method without needing a derivative. It requires two initial guesses and creates a line between the function values between of those guesses rather than a tangent line. The process then follows just as Newton's Method but with two approximation points at a time.]

Alternatively, the Secant Method exists for the same purpose as Newton's Method. It can also be used to find the roots of complex valued functions and it is competitive with Newton's Method. As mentioned in the further explanation of the definition box, while Secant Method requires two initial guesses, it's not necessary to have a derivative of the function and thus doesn't have the same issues at critical points that Newton's Method does. However, this project never examined a function that had an overly complicated derivative, or lacked a non-zero derivative. Further, the relative downsides of Secant Method were never examined. Thus, since no faults of Newton's Method were of consequence in our work, and it's marginally easier to use, we utilized Newton's Method.

Chapter 3

Real Valued Functions

Real Valued Functions:

$f(x) = y$ with $x, y \in \mathbb{R}$ [4][FE: Functions that operate in real numbers. Meaning, both the inputs (x) and the outputs (y) are real numbers. See introduction for further understanding of Real Numbers.]

3.1 Proof

We now prove Newton's Method Proof for Real Valued Functions. Let $f \in C^2[a, b]$ be such that $f(p) = 0$ and $f'(p) \neq 0$, then $\exists \varepsilon \ni \varepsilon > 0$ such that Newton's Method generates a sequence $\{p_n\}_{n=1}^{\infty}$ converging to p for any initial approximation $p_0 \in [p - \varepsilon, p + \varepsilon]$.

Proof: The proof is based on analyzing Newton's Method as the functional iteration scheme $p_n = g(p_{n-1})$, for $n \geq 1$, with

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

The object is to find, for a value k in $(0,1)$, an interval $[p-\varepsilon, p+\varepsilon]$ such that g maps the interval $[p-\varepsilon, p+\varepsilon]$ to itself and $|g'(x)| \leq k-1$ for $x \in [p-\varepsilon, p+\varepsilon]$. Since $f'(p) \neq 0$ and f' is continuous, $\exists \varepsilon_1 \ni f'(x) \neq 0$ for $x \in [p-\varepsilon_1, p+\varepsilon_1] \subset [a,b]$. Thus,$g$ is defined and continuous on $[p-\varepsilon_1, p+\varepsilon_1]$. Also,

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2}$$

for $x \in [p-\varepsilon_1, p+\varepsilon_1]$; since $f \in C^2[a,b]$ we have $g \in C^1[p-\varepsilon_1, p+\varepsilon_1]$. By assumption, $f(p) = 0$, so

$$g'(p) = \frac{f(p)f''(p)}{[f'(p)]^2} = 0.$$

Since g' is continuous, this implies that for any positive $k < 1$ $\exists$ an ε, with $0 < \varepsilon < \varepsilon_1$, and

$$|g'(x)| \leq k \text{ for } x \in [p-\varepsilon, p+\varepsilon].$$

It remains to show that $g : [p-\varepsilon, p+\varepsilon] \Rightarrow [p-\varepsilon, p+\varepsilon]$. If $x \in [p-\varepsilon, p+\varepsilon]$, the Mean Value Theorem implies that, for some number

α between x and p, $|g(X) - g(p)| = |g'(\alpha)||x - p|$. So

$$|g(x) - p| = |g(x) - g(p)| = |g'(\alpha)||x - p| \leq k|x - p| < |x - p|.$$

Since $x \in [p - \epsilon, p + \epsilon]$, it follows that $|x - p| < \epsilon$. This implies $g : [p - \epsilon, p + \epsilon] \Rightarrow [p - \epsilon, p + \epsilon]$. All the hypotheses of the Fixed-Point Theorem are now satisfied for $g(x) = x - \frac{f(x)}{f'(x)}$, so the sequence $\{p_n\}_{n=1}^{\infty}$ defined by

$$P_n = g(P_{n-1}), \text{ for } n = 1, 2, 3, \ldots,$$

converges to p for any $p_0 \in [p - \epsilon, p + \epsilon]$. [1] ❏

[FE: What this tells us is that Newton's Method "works" for real numbers. Meaning, when we have a function that uses real numbers, we can apply Newton's Method to it and find the roots.]

3.2 Examples

3.2.1 $f(x) = (x+1)(x-2)$

For the first real-valued function we will examine, using basic algebra skills, we can see that the roots of this function (where $f(x) = 0$) are $x = -1, 2$ i.e. at the points $(-1, 0)$ and $(2, 0)$ in the plane. We can also see this by looking at the graph of our function and noting where the function crosses the y-axis. [Note: All roots will have been determined before using Newton's Method in order to confirm the algorithm's accuracy as well as to enable us to better explore and observe behavior.]

In order to use Newton's Method we must first know the derivative. Recalling our calculus skills we find that $f'(x) = 2x - 1$. Now, we can use Newton's Method to find the roots. We won't be doing this by hand but rather using code written in Matlab; however, for reference recall the Newton's Method Theorem in the Background Knowledge section or Burden and Faires' Proof in the "Real Valued Functions" section. [Note: Just like manual Newton's Method, our code requires an epsilon value within which to stop the algorithm and an iteration limit, epsilon is .001 and iteration limit is 5,000 for all instances]

Remembering that Newton's Method requires an initial guess to start, let's guess something closer to the root $(-1, 0)$, suppose we enter

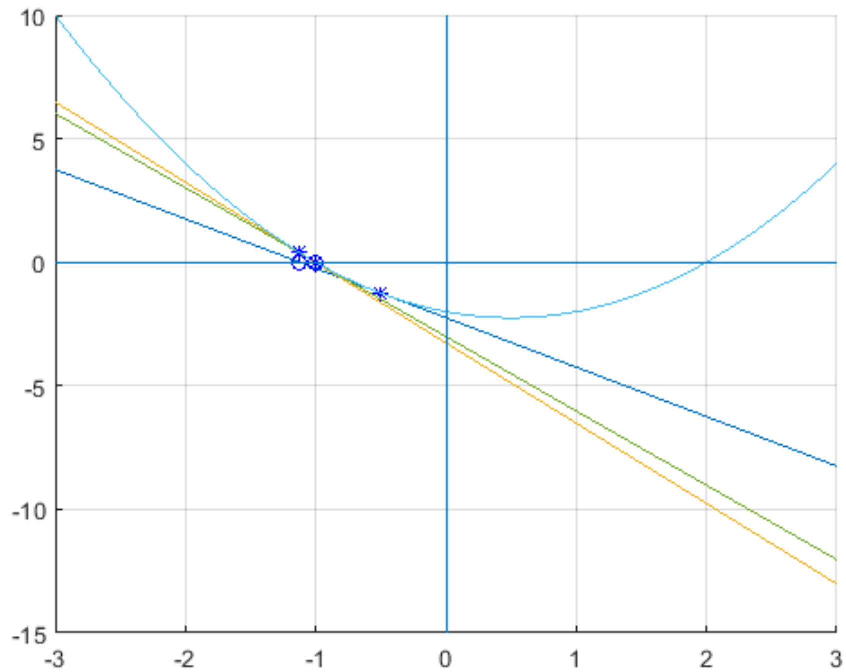

Figure 3.2.1: Newton's Method–quadratic polynomial

$x_0 = -0.5$. Then Newton's Method converges to -1 within 4 iterations. We can see this in the Figure 3.2.1.

As with all our later visuals similar to this, it shows a limited window of our function ($f(x) = (x+1)(x-2)$) as well as stars, open circles and lines. Each star is the function value of an x-intercept. Each open circle is an x-intercept (not including the initial guess). Each line is the tangent line for the corresponding function value (star it goes through) of an x-intercept (open circle the star correlates to). So, the way to read and interpret this visual requires knowing what the initial guess was (which always be given before the visual). In this case, our initial guess was at $x_0 = -0.5$ so we see the corresponding star and tangent line then the subsequent 4 iterations required to find the root. In this graphic we notice that the last two sets of the tangent lines, stars and circles are very closely bunched. This is due to our choice of epsilon. In this instance, the third iteration was close, but not epsilon close so there had to be another iteration which results in a crowded image.

Predictable Behavior (of a function):

(This is a unique definition for this paper.) Behavior of a function which is deterministic, it has no instabilities, no wildly unanticipated outcomes. [FE: When we throw a rock up it falls, when we throw it a little to the left it falls a little to the left. In our context, when we make an initial guess, we expected it to converge to its nearest root.]

Now, let's try an initial guess close to our other known root, let $x_0 = 1.5$. Once again, we got to our predicted known root, $x = 2$ and within 4 iterations too.

Since our initial guesses have been fairly close to our roots, we ought to try some much farther away. When $x_0 = -50$ we get the root $x = -1$ within 9 iterations and when $x_0 = 50$ we get the root $x = 2$ again within 9 iterations. Both of these results were predictable and normal since both initial guesses still converged to the closest root.

Next we will set our initial guess to an x-value equally distant to both points, $x_0 = 0.5$. Interestingly, our result is a lack of convergence. Examining possible reasons for this we ought to take note of the derivative at that point. $f'(0.5) = 2(0.5) - 1 = 0$. As mentioned previously, Newton's Method can't do anything with a derivative of 0 because it can't find another x-intercept to start another iteration after that (see further explanation in How Does Newton's Method Work? section elaborating on the algorithm).

x_0	converged to	in _ iterations
-0.5	-1	4
1.5	2	4
-50	-1	9
50	2	9
0.5	X	X

Table 3.2.1: Summary of Example 3.2.1

Thus, concerning the function $f(x) = (x+1)(x-2)$, we had a previous understanding (from algebra) of how this function would look and behave graphically, what the roots were and how to find them. There were only two unique, real roots, which allowed us the easiest example to begin understanding Newton's Method. When we used Newton's Method to find its roots, we found that initial guesses led to the closest root. Further, when an initial guess was equidistant to both roots, there was no convergence, coincidentally, because there was a slope of zero at that point. Ultimately, this function behaved predictably and we did not have to change any of our understandings in order to grasp what happened.

3.2.2 $f(x) = (x+2)(x-1)(x-4)$

For the second real-valued function we will examine, once more using our algebra skills, we can plainly see the roots of this function are $x = -2, 1, 4$ thus $(-2, 0), (1, 0), (4, 0)$, respectively. Remembering that Newton's Method needs a derivative we find that $f'(x) = 3(x^2 - 2x - 2)$. From here we examine how Newton's Method finds these roots. Let's make our initial guesses a value slightly more positive than the roots' x-values. So, $x_0 = 2$ gives us $x = 1$ within 4 iterations.

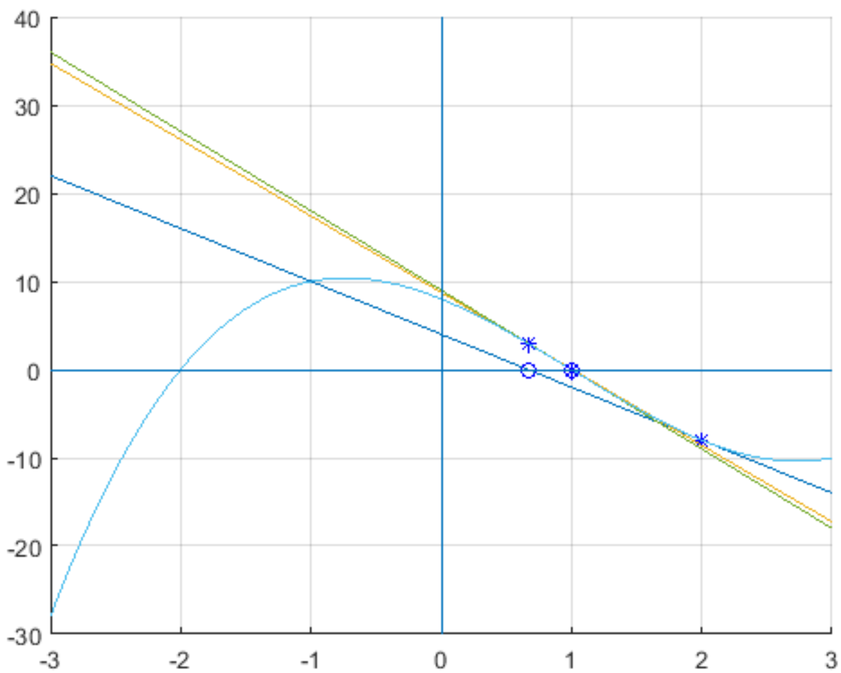

Figure 3.2.2: Newton's Method–cubic polynomial

An initial guess of $x_0 = -1$ gives us $x = -2$ within 6 iterations and an initial guess of $x_0 = 5$ gives us $x = 4$ within 4 iterations. This was all predictable behavior; initial guesses led to the nearest root. It is interesting to note the different rates of convergence, despite equal distance to the closest root.

We will now examine Newton's Method behavior when we use initial guesses with slightly more negative values than our root's x-values. So, $x_0 = -3$ gives us $x = -2$ within 4 iterations, $x_0 = 0$ gives us $x = 1$ within 4 iterations, and $x_0 = 3$ gives us $x = 4$ within 6 iterations. Again, it is interesting to note the speeds of convergence.

Now, we ought to examine points farther away from our roots. However, we will note that the root $(1, 0)$ is surrounded by the other two roots (one more positive, one more negative). Meaning, for far-off initial guesses, the other roots would always be the closer roots and thus $(1, 0)$ won't predictably have any far-reaching initial guesses lead to that root. So, letting $x_0 = -50$ we find the root $x = -2$ within 11 iterations. Then, letting $x_0 = 50$ we find the root $x = 4$ within 11 iterations. Finally, we will examine the x-values of points equidistant to different roots. With $x_0 = -0.5$ we get $x = 4$ within 2 iterations and with $x_0 = 2.5$ we get $x = -2$ within 2 iterations.

While these results seem like possibly unpredictable behavior, when examine the graph, making note of the function's symmetry, this behavior makes sense. In our last example the point equidistant to each of our known roots happened to be where the slope of the function was zero. In this case we see that the point equidistant between roots is not

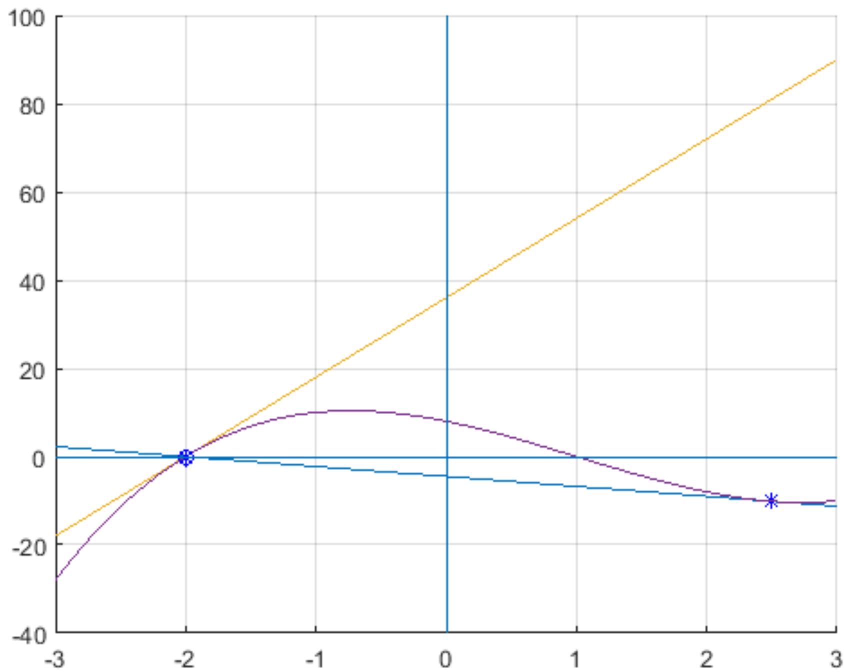

Figure 3.2.3: Newton's Method–

always a critical point (where the function's derivative is 0).

x_0	converged to	in _ iterations
2	1	4
-1	-2	6
5	4	4
-3	-2	4
0	1	4
3	4	6
-50	-2	11
50	4	11
-.5	4	2
2.5	-2	2

Table 3.2.2: Summary of Example 3.2.2

Thus, concerning the function $f(x) = (x+2)(x-1)(x-4)$, we, again, had a previous understanding (from algebra) of what this function would look like graphically, how it behaved graphically, what the roots were and how to find them. When we utilized Newton's

Method to find its roots, we found predictable behavior again, where initial guesses led to the closest root. Yet, with this function we were able to see that points equidistant between roots can behave oddly; rather than not converging, it converged to the farthest root (of the three roots in this problem). We can see that this is still not unpredictable behavior (rather, a symptom of symmetry) since moving the initial guess slightly more towards one root or the other would converge to that closer root. Ultimately, this function behaved predictably with a and we did not have to change our understanding in order to grasp what we were seeing.

3.2.3 $f(x) = \sin(x)$

For the last real-valued function we will examine, recalling our trigonometric knowledge, we recognize that this function is an oscillating function and therefore it technically has infinite roots. For our purposes, it should suffice to examine only one period of the sine function, so our roots are at $x = 0, \pi, 2\pi$.

Again, Newton's Method needs a derivative to function, our derivative is $f'(x) = \cos(x)$. Now we'll examine how Newton's Method finds these roots, working with our x-value in radians for easier comprehension. We'll set our initial guesses to a value slightly more positive than the root's x-values. So, $x_0 = \frac{\pi}{4}$ gives us $x = 0$ within 4 iterations.

[Note: Matlab is finding the root within the stopping tolerance epsilon, so the result is an answer that's approximated but "near-as-makes-no-difference."] With $x_0 = \frac{5\pi}{4}$ we get $x = \pi$ within 4 iterations and with

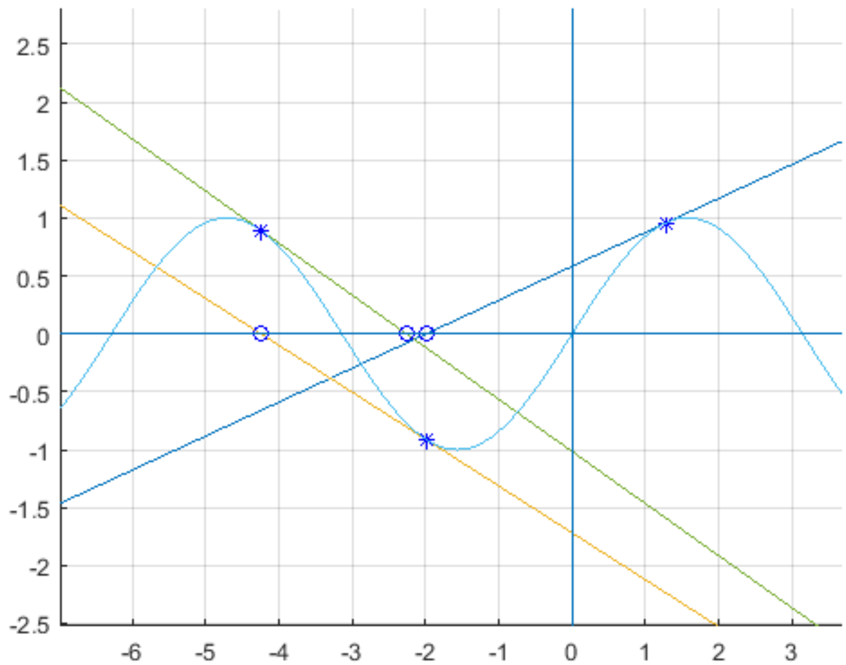

Figure 3.2.4: Newton's Method–sine

$x_0 = \frac{9\pi}{4}$ we get $x = 2\pi$ within 4 iterations. This is all predictable behavior since all initial guesses converging to the nearest root.

Next, we examine Newton's Method behavior when our initial guesses have slightly more negative values than the root's x-values. So, $x_0 = \frac{-\pi}{4}$ gives us $x = 0$ within 4 iterations. With $x_0 = \frac{3\pi}{4}$ we get $x = \pi$ within 4 iterations and with $x_0 = \frac{7\pi}{4}$ we get $x = 2\pi$ within 4 iterations. This is still all predictable behavior with all initial guesses converging to the nearest root.

As mentioned previously, the function is oscillating and there are infinite roots, therefore there's no use seeing the behavior when initial guesses are far from roots since there will always be another root near the guess. Nevertheless, we can examine behavior of points equidistant between roots. Yet, $x_0 = \frac{\pi}{2}$ does not converge (within 5000 iteration limit) and the same result is given for $x_0 = \frac{3\pi}{2}$. Going back to the first example of a real-valued function where the equidistant point happened to be critical points, this also is what happened for $f(x) = \sin(x)$ (recalling that for these roots are found with approximations of π). What's more, this will happen for every point equidistant between every root because of the oscillating nature.

x_0	converged to	in _ iterations
$\frac{\pi}{4}$	0	4
$\frac{5\pi}{4}$	π	4
$\frac{9\pi}{4}$	2π	4
$\frac{-\pi}{4}$	0	4
$\frac{3\pi}{4}$	π	4
$\frac{7\pi}{4}$	2π	4
$\frac{\pi}{2}$	X	X
$\frac{3\pi}{2}$	X	X

Table 3.2.3: Newton's Method–sine

Therefore, concerning the function $f(x) = sinx$, we had a previous understanding (from trigonometry) of what this function would look like graphically, how it behaved graphically, what the roots were and how to find them. When we utilized Newton's Method to find its roots, we found predictable behavior where initial guesses converged to the closest root but we had to take the functions oscillating nature into account. Further, when examining initial guesses equidistant to roots, it would always be at a critical point and thus never converge. It will be important later to remember that $f(x) = sinx$ is a real-valued function and in this case it also has real-valued inputs which result in real-valued outputs. Ultimately, this function behaved predictably known rate of convergence and we did not have to change our understanding in order to grasp what we were examining.

3.3 Analysis

We now understand how Newton's Method works and how it can be used as a tool to find the roots of a real-valued function. Further, we are now able to utilize Newton's Method as a tool for examining real-valued functions and the behavior of Newton's Method converging to the different roots of a function. After examining these three real-valued functions we are able to understand what predictable or normal behavior constitutes and are able to notice it in our algorithm results. Moreover, we now know that Newton's Method cannot converge when it encounters critical points and we know why. We're also being conscientious of the fact that critical points can, but do not always, fall on equidistant points between roots. Lastly, we noticed that Newton's Method can converge to unexpected roots but still not be exhibiting unpredictable/ chaotic behavior.

Chapter 4

Complex Valued Functions

4.1 How Does Newton's Method Work? -Complex

Complex Valued Functions:

$f(z) = c \ni z, c \in \mathbb{C}$ where $z, c = x + iy; x, y \in \mathbb{R}$ [4] [FE: Functions that operate in complex numbers. Meaning, the function operates using complex numbers but can have real-valued inputs and can have complex outputs with zero imaginary component. See introduction for further understanding of Complex Numbers.]

The way Newton's Method works does not change when we consider both Complex-valued functions nor Complex-valued inputs and outputs. Newton's Method is still an iterative algorithm that works in the same manner.

The only difference between Newton's Method working with Real-valued functions and Complex-valued functions is that we have (recalling from introduction) $f(z)$ where $z = x + iy \ni x, y \in \mathbb{R}$. This changes our understanding of inputs, outputs, the plane we are working in and can change our functions. (We will see that real-valued functions can be utilized in the complex-plane given complex-inputs and thus outputs.) It will prove important at this point to have an understanding of the complex plane (See overview).

4.2 Examples

4.2.1 $f(z) = z^3 - 1$

This function was extensively examined in the paper *Fractal Newton Basins* by M. L. Sahari and I. Djellit which heavily inspired and partially guided this project. As a result of having previous related work to investigate and compare to, this is the first equation that we comprehensively explored. The roots of this equation are $z = 1, -0.5 \pm i\frac{\sqrt{3}}{2}$. Though, since it is not intuitive nor common knowledge to find these roots, this project will outline how they are found. It is important to realize that roots are now at $(z, f(z))$ where $f(z) = 0$ and since $z = x + iy$, x and y-value are still significant for the purposes of graphing; thus visually, roots will be seen differently in the complex-plane.

As with finding the roots of any function we will set $f(z) = z^3 - 1 = 0$ and solve for z. First we see that

$$z^3 = 1,$$

then by De Moivre's Theorem we have that

$$z = r^3(\cos(3\theta) + i\sin(3\theta)) = 1 + i0,$$

Recalling theorems for converting rectangular form complex numbers to polar form complex numbers we have

$$r^3 = \sqrt{1^1 + 0^2} = 1 + i0$$

Splitting our equation into like-solutions we see that

$$\cos(3\theta) = 1 \text{ and } i\sin(3\theta) = 0i$$
$$\implies \theta = \frac{1}{3}\cos^{-1}(1) \text{ and } \theta = \frac{1}{3}\sin^{-1}(0)$$
$$\implies \theta = \frac{1}{3}(0, 2\pi, 4\pi, ..., 2n\pi) \text{ and } \theta = \frac{1}{3}(0, \pi, 2\pi, ..., n\pi) \ni n \in \mathbb{Z}$$
$$\implies \theta = \frac{1}{3}(0, 2\pi, ..., 2n\pi)$$
$$\implies \theta = 0, \frac{2\pi}{3}, ..., \frac{2n\pi}{3}$$
$$\implies z = 1\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\pm\frac{2\pi}{3}\right)$$
$$\implies z = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2} \text{ and } 1$$

Since we cannot generate an image of the function itself (being that this function maps $\mathbb{R}^2 \to \mathbb{R}^2$ so that would be a fourth dimension visual) we have provided a very specified point-wise Newton Basin graphic. To read this, and our similar visuals later, first note the roots which are $z_1 = 1$, $z_2 = \frac{-1}{2} - \frac{\sqrt{3}}{2}$, $z_3 = \frac{-1}{2} + \frac{\sqrt{3}}{2}$. Then simply see that, as labelled, the blue region or dots show that an initial guess at a point in that region will converge to the labelled root (i.e. each blue dot is an initial guess that converged to that specified root). It's important to realize that for this example the paper *Fractal Newton Basins* provided a similar visual but for each of the complex-valued functions we explored (including this first function), we did not examine the Newton Basins until we had already revealed specific point-wise examples of Chaotic behavior. Thus, this image logically belongs here to provide us a clear visual of what we were exploring but it, and the proceeding visuals, were not used to direct that exploration.

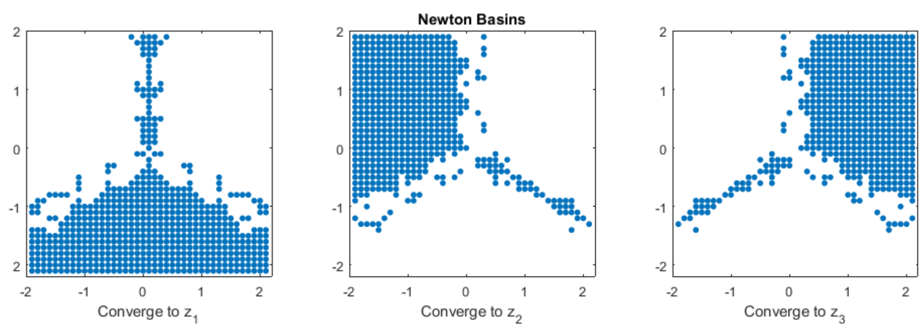

Figure 4.2.1: Newton Basins for $z^3 - 1$

As we know, Newton's Method still needs the derivative to work, deriving complex-valued functions is done in the same manner as with real-valued functions since the only real difference is the presence of $i = \sqrt{-1}$ which is only a number, not a variable. Therefore we have that $f'(z) = 3z^2$.

As with our real-valued function examples let's examine the behavior of Newton's Method with initial guesses slightly more positive z-values than the z-value of our roots. To do this we will change both the x and y-value, to change the z-value. So, let $z_0 = 1.3 + i(0.3)$, this gives us the root $z = 1 + i0$ within 5 iterations.

Let $z_0 = -0.2 + i(\frac{\sqrt{3}}{2} + 0.3)$, this gives us the root $z = -0.5 + i\frac{\sqrt{3}}{2}$ within 5 iterations.

Let $z_0 = -0.2 - i(\frac{\sqrt{3}}{2} - 0.3)$, this gives us the root $z = -0.5 - i\frac{\sqrt{3}}{2}$ within 5 iterations.

This is all predictable/ normal behavior since initial guesses are converging to the nearest root. Now we will examine if this behavior continues with initial guesses that have slightly more negative z-values than the z-values of our roots. Let $z_0 = 0.7 - i(0.3)$, this gives us the root $z = 1 + i0$ within 6 iterations.

Let $z_0 = -0.8 + i(\frac{\sqrt{3}}{2} - 0.3)$, this gives us the root $z = -0.5 + i\frac{\sqrt{3}}{2}$ within 6 iterations.

Let $z_0 = -0.8 - i(\frac{\sqrt{3}}{2} + 0.3)$, this gives us the root $z = -0.5 - i\frac{\sqrt{3}}{2}$ within 5 iterations. All predictable/ normal behavior, initial guesses are converging to the nearest root.

Unpredictable/ Chaotic Behavior (of a function):

(this is a unique definition for this paper) Behavior of a function where small changes in in initial conditions (the problem settings) result in wildly different outcomes. (This concept can exist and be observed in different ways, we can have a resultant qualitatively different answer–not just quantitatively.) [F.E.: When we throw a rock up it falls, when we throw it a little to the left it falls a way behind us.]

Next we will examine if this predictable behavior continues with initial guesses that have a much more positive z-value than the z-value of our roots, followed by much more negative z-value than the z-value of our roots. Let $z_0 = 5 + i5$, this gives us the root $z = 1 + i0$ within 11 iterations.

Let $z_0 = 3 + i3(\frac{\sqrt{3}}{2})$, this gives us the root $z = 1 + i0$ within 9 iterations.

Let $z_0 = 3 - i0$, this gives us the root $z = 1 + i0$ within 7 iterations.

Let $z_0 = -4 - 4i$, this gives us the root $z = -0.5 - i\frac{\sqrt{3}}{2}$ within 9 iterations.

Let $z_0 = -3 + i0)$, this gives us the root $z = 1 + i0$ within 10 iterations.

Let $z_0 = -3 - i3(\frac{\sqrt{3}}{2})$, this gives us the root $z = -0.5 - i\frac{\sqrt{3}}{2}$ within 8 iterations. This behavior has become interesting, though according to our definition it is still not chaotic behavior, we are beginning to see Newton Basins where initial guesses within certain regions of the complex plane converge to certain roots. What is more intriguing is what behavior might occur when we get close to the border between two different regions (i.e. One region may converge to 1 while the region next to it converges to $-0.5 - i\frac{\sqrt{3}}{2}$, we are interested in what happens along the border between these two regions.) This is what we will explore next.

We can roughly see where there are "borders" between these regions (See Newton Basin Visuals). When we look closely at z-values around $z_0 = 0.5 + i\frac{\sqrt{3}}{2}$ we can see chaotic behavior since very small changes create wildly different outcomes: When $z_0 = 0.5 + i\frac{\sqrt{3}}{2}$ we are returned a root of $z = -0.5 - i\frac{\sqrt{3}}{2}$ within 9 iterations, we are returned the same root with $z_0 = 0.5 + (1 + 0.304)i\frac{\sqrt{3}}{2}$ within 18 iterations.

Yet, when we use $z_0 = 0.5 + (1 + .303)i\frac{\sqrt{3}}{2}$ we are given the root $z = -0.5 + i\frac{\sqrt{3}}{2}$ within 15 iterations.

Further, when we use $z_0 = 0.5 + (1 + 0.305)i\frac{\sqrt{3}}{2}$ we are given the root $z = 1 + i0$ within 16 iterations.

This is chaotic behavior. With a change of only $(0.001)i$ Newton's Method converges to either of the different roots and if we were so compelled we could determine the even smaller changes necessary to converge to the different roots.

z_0	converged to	in _ iterations
$z_0 = 1.3 + i(0.3)$	$1 + i0$	5
$z_0 = -0.2 + i(\frac{\sqrt{3}}{2} + 0.3)$	$-0.5 + i\frac{\sqrt{3}}{2}$	5
$z_0 = -0.2 - i(\frac{\sqrt{3}}{2} - 0.3)$	$-0.5 - i\frac{\sqrt{3}}{2}$	5
$z_0 = .7 - i(.3)$	$1 + i0$	6
$z_0 = -0.8 + i(\frac{\sqrt{3}}{2} - 0.3)$	$-0.5 + i\frac{\sqrt{3}}{2}$	6
$z_0 = -0.8 - i(\frac{\sqrt{3}}{2} + 0.3)$	$-0.5 - i\frac{\sqrt{3}}{2}$	5
$z_0 = 5 + i5$	$1 + i0$	11
$z_0 = 3 + i3(\frac{\sqrt{3}}{2})$	$1 + i0$	9
$z_0 = 3 - i0$	$1 + i0$	7
$z_0 = -4 - 4i$	$-0.5 - i\frac{\sqrt{3}}{2}$	9
$z_0 = -3 + i0)$	$1 + i0$	10
$z_0 = -3 - i3(\frac{\sqrt{3}}{2})$	$-0.5 - i\frac{\sqrt{3}}{2}$	8
$z_0 = 0.5 + i\frac{\sqrt{3}}{2}$	$-0.5 - i\frac{\sqrt{3}}{2}$	9
$z_0 = 0.5 + (1 + 0.304)i\frac{\sqrt{3}}{2}$	$-0.5 - i\frac{\sqrt{3}}{2}$	18
$z_0 = 0.5 + (1 + 0.303)i\frac{\sqrt{3}}{2}$	$-0.5 + i\frac{\sqrt{3}}{2}$	15
$z_0 = 0.5 + (1 + 0.305)i\frac{\sqrt{3}}{2}$	$1 + i0$	16

Table 4.2.1: Newton's Method–complex cubic

Newton Basins/ Basins of Attraction:

"A Newton basin is just the set of initial guesses that lead to one solution or root" [3]. [FE: When a function has roots, using Newton's Method we can make an initial guess to find those roots. As we will see in the Real Valued Function section, each initial guess has the potential to lead to the root of the function. However, each initial guess has the potential to lead to a different root or even no root. Thus, Basins of Attraction are areas on the graph determined by which initial guesses lead to what root.]

Following are the more "zoomed out" Newton Basins that showcase the chaotic nature and fractal structure. At this point we can realize that the border areas of each region is where we see chaotic behavior.

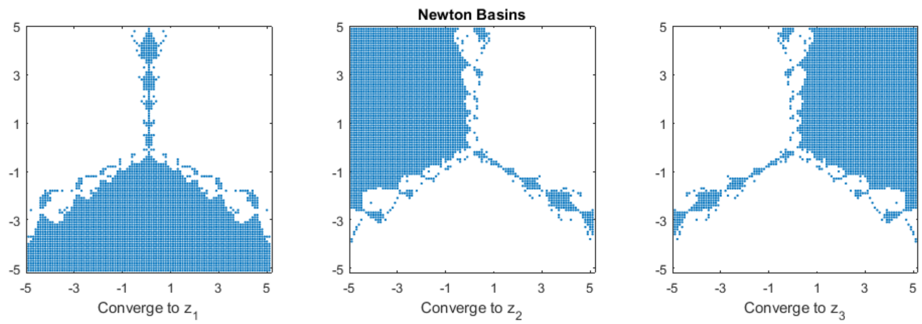

Figure 4.2.2: Newton's Method–complex cubic revisited

As mentioned earlier, because this portion of this project was guided by the paper *Fractal Newton Basins*, discussion of comparisons is obligatory. The lens of this project is relatively the same as the paper; both pieces are examining the behavior of complex-valued functions in the context of Newton's Method, in addition to exploring the resultant existence of Newton Basins. Although, *Fractal Newton Basins* had more in-depth exploration and conversation on Newton Basins and border behavior, this project has a more robust examination of the function itself. This project first studies Newton's Method in terms of real-valued functions but more significantly by exploring both the real and imaginary components of the complex-valued functions explored. *Fractal Newton Basins* on the other, had separated the real and imag-

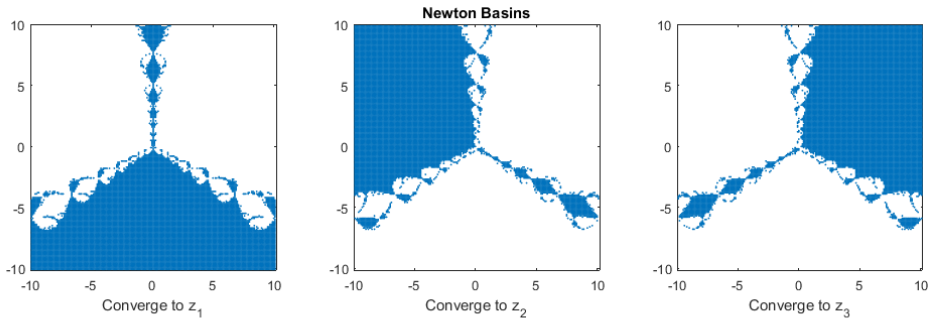

Figure 4.2.3: Newton's Method–complex cubic revisited again

inary components of $f(z) = z^3 - 1$ in order to explore the behavior of the function. (It's worth mentioning that the paper did not specify what tool was used for calculations, nor did they indicate the level of accuracy they did their calculations with.)

Besides having previous vetted work to compare with, this function was used because it had finite roots and was thus an easy complex-valued function to use as the first in order to have an optimal progression of understanding. As we explained, complex-numbers, functions and the plane must all be thought of differently from that of real-numbers. It's best to begin forming a comprehension with a function that can be sufficiently inspected within a certain window. Finally, this was the first and simplest instance of chaotic behavior that we were able to see and explore. It's significant to note that the above initial guesses that resulted in chaotic behavior are not the single and only occurrence within for this function. Based on the observed border behavior, by looking at the Newton Basin visual for this function, one can conceptualize other similar behavior. To preserve the purpose of this project and for brevity's sake, this project cannot explore all potential chaotic behavior of each function, but rather analyze the cases that we do notice.

4.2.2 $f(z) = e^z - 1$

A natural extension from the previous complex-valued function, this function has infinite roots. Similar to the last function, we will limit our view of the function, however, this time it will be clear since it's necessary in order to closely explore the behavior converging to a few of the many roots. These roots are $z = (x + iy)$ such that $x = 0, y = 2k\pi$ where $k \in \mathbb{Z}$ (meaning that there's a root where x is 0 and y is some even number of π, including 0). Since determining these roots is not intrinsic nor common knowledge, the process by which these were found is as follows:

Setting $f(z) = 0$ to solve for z we start with

$$e^z - 1 = 0$$

$$\implies e^{x+iy} - 1 = 0$$

$$\implies e^x e^{iy} = 1$$

Using the complex exponential function theorem

$$\implies e^x(\cos(y) + i\sin(y)) = 1 + i0$$

$$\implies e^x\cos(y) + e^x i\sin(y)) = 1 + i0$$

Splitting the equation into like-solutions we have

$$\implies e^x\cos(y) = 1 \text{ and } e^x i\sin(y) = i0$$

$$\implies \sin(y) = 0$$

$$\implies y = k\pi \ni k \in \mathbb{Z}$$

$$\implies e^x\cos(k\pi) = 1 \ni k \in \mathbb{Z}$$

$$\implies e^x(1) = 1 \text{ when } k \text{ is even and } e^x(-1) = 1 \text{ when } k \text{ is odd}$$

$$\implies x = \ln(1) \text{ when } k \text{ is even and } x = \ln(-1) \text{ when } k \text{ is odd}$$

Since $ln(-1)$ does not exist (DNE) we have that

$$x = 0 \text{ when } k \text{ is even and } x \text{ DNE when } k \text{ is odd}$$

$$\therefore z = x + iy \ni x = 0, y = 2k\pi \ni k \in \mathbb{Z}$$

For the provided visuals of the Newton Basin for this function, note that the roots are labelled as follows: $z_1 = 0$, $z_2 = -2i\pi$, $z_3 = 2i\pi$, $z_4 = 4i\pi$, $z_5 = -4i\pi$

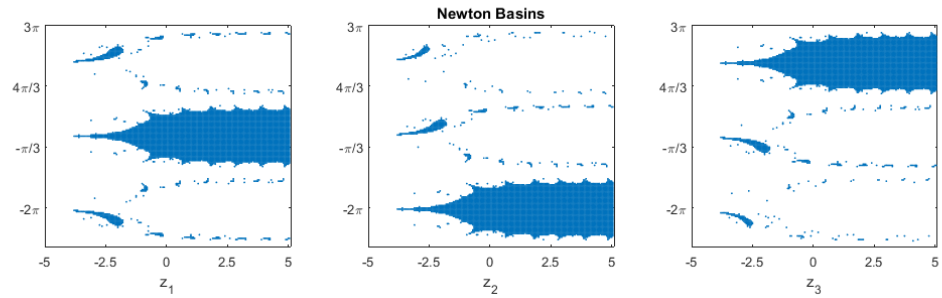

Figure 4.2.4: Newton's Method–complex exponential

Knowing Newton's Method needs the derivative of the function, we compute it to be $f'(z) = e^z$. We will limit our view of the function to two of the infinite roots, being $(0 + i0, 0)$ and $(0 + i2\pi, 0)$. Now, we will explore the behavior of Newton's Method convergence to roots when our initial guesses have z-values slightly more positive then more negative, respectively, of the z-values of our roots. Let $z_0 = .2 + i(.2)$, we converge to $z = 0 + i0$ within 4 iterations.

Let $z_0 = 0.2 + i(2\pi + 0.2)$, we converge to $z = 0 + i2\pi$ within 4 iterations.

Let $z_0 = -0.2 - i(0.2)$, we converge to $z = 0 + i0$ within 4 iterations.

Let $z_0 = -0.2 + i(2\pi - 0.2)$, we converge to $z = 0 + i2\pi$ within

4 iterations.

Each of these initial guesses led Newton's Method to showcase predictable convergence behavior, i.e. each guess converged to the nearest root. It's interesting to note that they each required the same number of iterations to converge, in other words, they all had the same speed of convergence.

Next, let's explore the behavior of Newton's Method convergence to roots when our initial guesses have z-values much more positive then more negative, respectively, of the z-values of our roots. Let $z_0 = 1.5 + i(\frac{7\pi}{8})$, we converge to $z = 0 + i81.681$ within 135 iterations.

Let $z_0 = 1.5+i(1.5)(2\pi)$, we fail to converge because of a division by zero.

Let $z_0 = 5+i(1.5)(2\pi)$ (since x-values shouldn't change outcome), we fail to converge because of a division by zero.

Let $z_0 = -1.5 - i(1.5)$, we failed converge within 5000 iterations.

Let $z_0 = -5 - i(1.5)$ (since x-values shouldn't change outcome), we converge to $z = 0 + 144.513$ within 13 iterations.

Let $z_0 = -1.5 + i(\frac{9\pi}{8})$, we fail to converge because of a division by zero.

Every initial condition resulted in wildly different and unpredictable results, this will not be classified as chaotic behavior in and of itself since as yet it needs to be determined if epsilon close points result in wildly different results as well. However, by doing so we can, in fact, see chaotic behavior. Letting $z_0 = 1.5 + i(1.5)(2\pi)$, we saw no convergence; letting $z_0 = 1.5 + i(1.478)(2\pi)$, we saw no convergence, However, letting $z_0 = 1.5 + i(1.477)(2\pi)$ we converged to $0 + i(-716.283)$ within 45 iterations. Further, letting $z_0 = 1.5 + i(1.476)(2\pi)$ we converged to $0 + i(-697.433)$ within 90 iterations. This can be classified as chaotic behavior since only small changes brought on wildly different outputs.

We will lastly examine the behavior with the initial guess at a point equidistant between the two roots we're looking at. Letting $z_0 = 0 + i\pi$, Newton's Method fails to converge due to a division of 0. However, examining the region around this equidistant point we observe more interesting behavior: letting $z_0 = .01 + (1 + .1008)i\pi$, we converge to $z = 0 + -5013.981875$ within 267 iterations letting $z_0 = .01 + (1 + .1007)i\pi$, we converge to $z = 0 + -5026.548246$ within 285 iterations letting $z_0 = .01 + (1 + .1009)i\pi$, we converge to $z = 0 + -5013.981875$ within 272 iterations Here we see chaotic behavior, a difference of only $.0001\pi$ caused convergence to completely different roots, both of which are not nearly the closest root to the initial guess point but rather roughly 798.24 roots away considering the repeating pattern of the roots.

z_0	converged to	in _ iterations
$z_0 = .2 + i(.2)$	$0 + i2\pi$	4
$z_0 = -.2 - i(.2)$	$0 + i0$	4
$z_0 = -.2 + i(2\pi - .2)$	$0 + i2\pi$	4
$z_0 = 1.5 + i(\frac{7\pi}{8})$	$0 + i81.681$	135
$z_0 = 1.5 + i(1.5)(2\pi)$	X	X
$z_0 = 5 + i(1.5)(2\pi)$	X	X
$z_0 = -1.5 - i(1.5)$	X	X
$z_0 = -5 - i(1.5)$	$0 + 144.513$	13
$z_0 = -1.5 + i(\frac{9\pi}{8})$	X	X
$z_0 = 1.5 + i(1.5)(2\pi)$	X	X
$z_0 = 1.5 + i(1.478)(2\pi)$	X	X
$z_0 = 1.5 + i(1.477)(2\pi)$	$0 + i(-716.283)$	45
$z_0 = 1.5 + i(1.476)(2\pi)$	$0 + i(-697.433)$	90
$z_0 = .01 + (1 + .1008)i\pi$	$0 + -5013.981875$	267
$z_0 = .01 + (1 + .1007)i\pi$	$0 + -5026.548246$	285
$z_0 = .01 + (1 + .1009)i\pi$	$0 + -5013.981875$	272

Table 4.2.2: Summary of 4.2.2

The second and third images are highlighting the regions where initial guesses fail to converge.

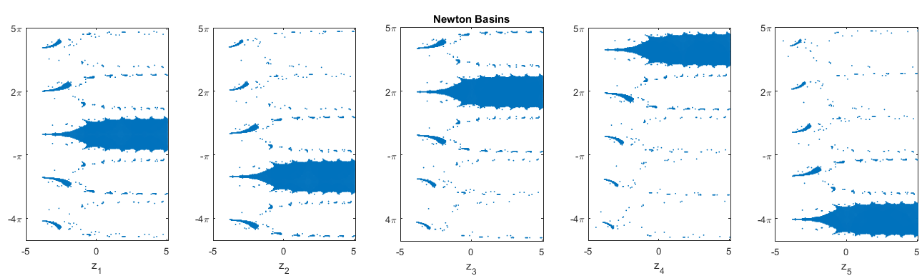

Figure 4.2.5: Newton's Method–expanded

Ultimately, we found some evidence of actual chaotic behavior, according to our definition, despite seeing very odd results. Further exploration is necessary to find more chaotic behavior, though, because

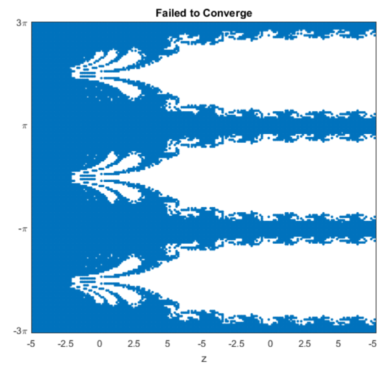

Figure 4.2.6: Newton's Method–failed

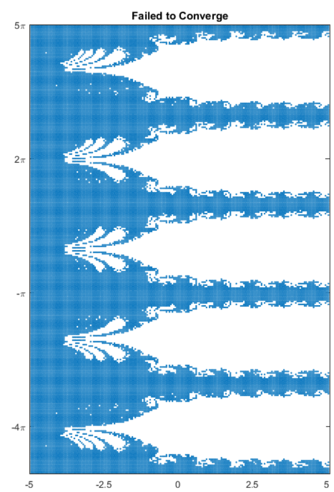

Figure 4.2.7: Newton's Method–failed zoom

of the functions repetitive nature, it's likely that we could easily examine similar behavior in another "repetition." It's interesting that with infinite roots it was more difficult to find chaotic behavior. There may be something to be said about the symmetry of the previous functions and the resulting behavior we were able to identify. In the last function, the three roots were equidistant from one another in a circular form, while this function has inherent limits since x can never be non-zero and were equidistant and repetitious in a linear manner. The next function, which has repetitious behavior too, may be able to give us further perspective on this.

4.2.3 $f(z) = \sin(z)$

This function is an exploration inspired from our third real-valued function example. Just as with the last complex-valued function, this one is infinite with infinite roots. Just as the real-valued $f(x) = \sin(x)$, this function exhibits that infinite oscillating manner. The roots of this function are when $iy = 0$ and $x = \sin^{-1}(0) = 0, \pi, ..., k\pi \ni k \in \mathbb{Z}$. Once more, since solving the roots of this function is not common knowledge the process we used is as follows:

Setting $f(z) = \sin(z)$ to zero to find our roots we get

$$\sin(z) = 0$$

From the complex exponential function theorem we have

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{4.1}$$

$$e^{-i\theta} = \cos(\theta) - i\sin(\theta) \tag{4.2}$$

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{4.3}$$

Now taking (4.1) and (4.2) we get

$$sin(\theta) = \tfrac{e^{i\theta} - e^{-i\theta}}{2i} \text{ as seen in (4.3)}$$
$$\implies sin(z) = \tfrac{e^{iz} - e^{-iz}}{2i}$$

and from our initial statement we have

$$\tfrac{e^{iz} - e^{-iz}}{2i} = 0$$

$$\implies e^{iz} - e^{-iz} = 0$$

$$\implies e^{iz} = e^{-iz}$$

$$\implies e^{i(x+iy)} = e^{-i(x+iy)}$$

$$\implies i(x+iy) = -i(x+iy)$$

$$\implies ix - y = -ix + y$$

$$\implies 2ix = 2y$$

$$\implies ix = y \ni x, y \in \mathbb{R}$$

$\therefore$ the only solution is when $x = y = 0$

However, recalling this is a sinusoidal function, there must be infinite roots, which are repeating in the periodic manner of sine.

Looking for our roots by examining

$$\sin(z) = 0$$

$$\implies \sin(x+iy) = 0 + i0$$

$$\implies x + iy = \sin^{-1}(0 + i0)$$

$$\implies x + iy = 0 + i0, \pi + i0, ..., k\pi + i0 \ni k \in \mathbb{Z}$$

$$\therefore iy = 0 \text{ and } x = \sin^{-1}(0) = 0, \pi, ..., k\pi \ni k \in \mathbb{Z}$$

Once more, the above Newton's Method visual has the following roots: $z_1 = 0$, $z_2 = -\pi$, $z_3 = \pi$, $z_4 = -2\pi$.

Since Newton's Method needs a derivative we find it to be $f(z) = cos(z)$. We will limit our view of the function to two of the infinite roots, being $(0 + i0, 0)$ and $(\pi + i0, 0)$. It should be noted that this

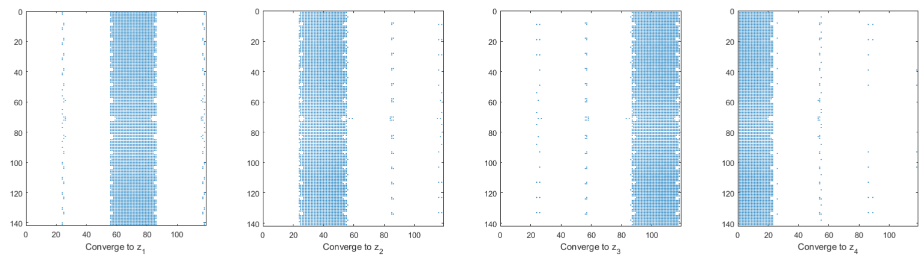

Figure 4.2.8: Newton's Method–complex sine

function is examined far more than all the previous functions. This primarily due to the fact that examining points equidistant between the two roots shows particularly interesting convergence behavior and speeds of convergence. For ease of understanding, this section's point-wise analysis and computational results will be condensed into the table (4.2.3) shown below. Similar discussion of these results, as seen in all other previous sections will still be available, simply with allusions to the table rather than a demonstration of the computations themselves.

Now, we will explore the behavior of Newton's Method convergence to roots when our initial guesses have z-values slightly more positive than the z-values of our roots. In considering $z_0 = \frac{\pi}{64} + i0$, $z_0 = (\pi + \frac{\pi}{64}) + i0$, $z_0 = \frac{\pi}{64} + (\frac{\pi}{64})i$, and $z_0 = (\pi + \frac{\pi}{64}) + (\frac{\pi}{64})i$ see rows 2-5 in the table (4.2.3).

Next we examine the behavior of the algorithm convergence to roots when our initial guesses have z-values slightly negative from the z-values of our roots. Considering when $z_0 = -\frac{\pi}{64} - i0$, $z_0 = (\pi - \frac{\pi}{64}) - i0$, $z_0 = -\frac{\pi}{64} - (\frac{\pi}{64})i$, and $z_0 = (\pi - \frac{\pi}{64}) - (\frac{\pi}{64})i$, see rows 6-9 in the table (4.2.3).

So far, this is all very appropriate and predictable convergence behavior; each initial z-value resulted in Newton's Method converging to the nearest root. We must now explore farther-off initial z-values, beginning with initial z-values much more positive than the z-values of our roots. For when $z_0 = (\frac{\pi}{2} - \frac{\pi}{64}) + i0$, $z_0 = (\frac{3\pi}{2} - \frac{\pi}{64}) + i0$, $z_0 = (\frac{\pi}{2} - \frac{\pi}{64}) + 1i$, and $z_0 = (\frac{3\pi}{2} - \frac{\pi}{64}) + 1i$, see rows 10-13 in the table (4.2.3).

Clearly we must now examine the convergence of Newton's Method when the initial z-value is much more negative than the z-values of our roots. For $z_0 = (-\frac{\pi}{2} + \frac{\pi}{64}) + i0$, $z_0 = (\frac{\pi}{2} + \frac{\pi}{64}) + i0$, $z_0 = (-\frac{\pi}{2} + \frac{\pi}{64}) - 1i$, $z_0 = (\frac{\pi}{2} + \frac{\pi}{64}) - 1i$, see rows 14-17 in the table (4.2.3).

Finally we ought to explore the convergence of Newton's Method when we test initial z-values that are equidistant between our two roots.

Letting $\frac{\pi}{2} + i0$, we converge to $z = -16331239353195368 + i0$ within 4 iterations (see row 18 of the table).

This was a very interesting result, however, we cannot yet determine if this is chaotic behavior though it's certainly not an intuitive result. So, let's explore the behavior of Newton's Method given initial

z-values that are surrounding the equidistant point $z_0 = \frac{\pi}{2}$. See the remaining rows 19-28 in the table (4.2.3) for the initial guesses tested and the computational results that followed.

z_0	converged to	in _ iterations
$z_0 = \frac{\pi}{64} + i0$	$z = 0 + i0$	3
$z_0 = (\pi + \frac{\pi}{64}) + i0$	$z = \pi + 0i$	3
$z_0 = \frac{\pi}{64} + (\frac{\pi}{64})i$	$z = 0 + i0$	3
$z_0 = (\pi + \frac{\pi}{64}) + (\frac{\pi}{64})i$	$z = \pi + 0i$	3
$z_0 = -\frac{\pi}{64} - i0$	$z = 0 + i0$	3
$z_0 = (\pi - \frac{\pi}{64}) - i0$	$z = \pi + 0i$	3
$z_0 = -\frac{\pi}{64} - (\frac{\pi}{64})i$	$z = 0 + i0$	3
$z_0 = (\pi - \frac{\pi}{64}) - (\frac{\pi}{64})i$	$z = \pi + 0i$	3
$z_0 = (\frac{\pi}{2} - \frac{\pi}{64}) + i0$	$z= -18.850 + i0	4
$z_0 = (\frac{3\pi}{2} - \frac{\pi}{64}) + i0$	$z = -15.708 + i0$	4
$z_0 = (\frac{\pi}{2} - \frac{\pi}{64}) + 1i$	$z = 0 + i0$	8
$z_0 = (\frac{3\pi}{2} - \frac{\pi}{64}) + 1i$	$z = \pi + i0$	8
$z_0 = (-\frac{\pi}{2} + \frac{\pi}{64}) + i0$	$z = 18.850 + i0$	4
$z_0 = (\frac{\pi}{2} + \frac{\pi}{64}) + i0$	$z = 21.991 + i0$	4
$z_0 = (-\frac{\pi}{2} + \frac{\pi}{64}) - 1i$	$z = 0 + i0$	8
$z_0 = (\frac{\pi}{2} + \frac{\pi}{64}) - 1i$	$z = \pi + i0$	8
$\frac{\pi}{2} + i0$	$z = -16331239353195368 + i0$	4
$\frac{\pi}{2} + 1i$	$z = 0 + i0$	78
$\frac{\pi}{2} - 1i$	$z = 0 + i0$	78
$(\frac{\pi}{2})(1 + .0001) + 0i$	$z = 6368.008309 + i0$	4
$(\frac{\pi}{2})(1 + .0002) + 0i$	$z = 3185.574951 + i0$	6
$(\frac{\pi}{2})(1 + .0003) + 0i$	$z = 2123.716634 + i0$	4
$(\frac{\pi}{2})(1 + .0003) + (.1)i$	$z = \pi + i0$	18
$(\frac{\pi}{2})(1 + .0003) + (.01)i$	$z = 2\pi + i0$	104
$(\frac{\pi}{2})(1 + .0003) + (.001)i$	X	X
$(\frac{\pi}{2})(1 + .0003) + (.0019)i$	$z = 125.663706 + i0$	501
$(\frac{\pi}{2})(1 + .0003) + (.002)i$	$z = 113.097336 + i0$	478

Table 4.2.3: Newton's Method–complex sine

This is certainly all chaotic behavior, seemingly any small change resulted in wildly different convergences. Thus, this function $f(z) = \sin(z)$ has showcased much chaotic behavior. Interestingly, we found a lot of chaotic behavior surrounding the point equidistant between two roots that we had focused our observation on. This observation was not particularly unique to this function, though we were able to find ample chaotic behavior in the equidistant region without much searching at all. Recall that $f(z) = e^z - 1$ exhibited chaotic behavior at equidistant points and though the function did not demonstrate sinusoidal behavior.

4.3 Analysis

From examining these three complex-valued functions, we were able to see and understand how Newton's Method operates when working with complex-valued functions compared to real-valued functions. Further, we understand that Newton's Method can showcase chaotic convergence behavior when in this complex context and we know what it means as well as how to observe this. What's more, we observed odd or unexpected convergence for complex valued functions that was not chaotic behavior, similar to the odd behaviors we saw with real-valued functions. Additionally, while we already understood that we can run into non-convergence issues with real-valued functions, after examining these functions we understand that the same situations are possible for complex valued functions. Specifically, we saw division of near zero errors and iteration limit errors of non-convergence. We were even able to see in our Newton Basin visual for $e^z - 1$ with entire areas/ regions of non-convergence. As a whole, we can note that we observed chaotic behavior for each of our complex valued functions, most of which was not unique, simply meaning we saw chaotic behavior at equidistant points and, logically, saw chaotic behavior along the region boarders of Newton Basins.

Chapter 5

Conclusion

5.1 Conclusive Analysis

In the end, we explained what our readers needed to know and understand in order to grasp the content of this project, and we defined some of the necessary terms. Next, we explained what Newton's Method is and we showcased the proof proof for real valued functions. We then examined the behavior of three real-valued functions when we used the iterative algorithm that is Newton's Method to find their roots. Through this, we gained an understanding of what predictable behavior in the context of Newton's Method convergence was and how to identify it as well as the significance of critical points of a function in Newton's Method. After examining all of our real-valued function examples, we were able to realize that real-valued functions do not exhibit chaotic convergence behavior. From here we looked at the proof for Newton's Method working with complex-valued functions, and explained this new context. We explored three complex-valued functions and established that the some of the same patterns that applied to real valued functions apply to complex valued functions, namely: issues of non-convergence such as at critical points, and odd convergence behavior (which could be predictable or chaotic for complex valued-functions) at points equidistant between roots. We were even able to create a visual for Newton Basins that revealed entire areas of non-convergence. We have learned what constitutes chaotic behavior as well as how to identify it in our convergence behavior. Then, those same visuals also allowed us to realize the next significant difference for complex valued functions, which was that the border regions of Newton Basins were areas of chaotic behavior. Moreover, this was a non-unique characteristics of chaotic convergence which was illustrated with and throughout each of our complex valued functions.

Broadly, this project was able to showcase how Newton's Method convergence can be used to identify interesting behavior for both real and complex valued functions. This was accomplished by defining behavior as particular point-wise convergence. Then, identifying the roots of the function and testing initial guesses, point by point, we found and revealed the structure of the convergence behavior for each of our example functions. What we are able to walk away from this project with is firstly, a comprehensive understanding of complex functions and fractals that are produced using our tool. Secondly, that it appears only but so far all complex-valued functions, regardless of function class, can demonstrate chaotic convergence behavior. And, finally, though we did not find a "distinguishing factor that determines whether behavior with be predictable or chaotic" (besides a function being real or complex); we still created the one of very few comprehensive studies of an iterative method in the context of solving complex valued equations.

5.2 Future Work

After the accumulation of our exploration and analysis, there are questions left unanswered and minutia that could prove interesting to explore further. First and foremost is, researching as to whether there are complex functions that will not showcase any chaotic convergence behavior. This was mentioned in the abstract as an intent of our project. However, the task at hand proved more complicated than expected and time did not allow for this to be discovered, let alone explored in-depth. Finding such behavior would be unique amongst our other work since, obviously, all of our chosen complex functions showcased chaotic behavior. This uniqueness is important because it effects any discussion of implications and patterns that we might have found in this work.

Correspondingly, the abstract also mentioned there would be an attempt to “find some distinguishing factor that determines whether behavior will be predictable or chaotic.” Had we explored this and had we seen a complex-valued function that displayed no chaotic behavior, the effort required to identify any “distinguishing factor” would have increased exponentially. Given more time and resources we would scrutinize far more complex-valued functions to determine if there are any identifiable characteristics that could allow us to typify whether or not they will exhibit chaotic behavior. Further, we could research if there are recurring fractal patterns among those that display chaotic behavior. This potential work is significant because it would be unique to this area of mathematics. Mathematical chaos itself, as mentioned very early on, has no universal unambiguous definition. So, being able to establish any sort of order in the subject of mathematical chaos would be significant work.

Beyond that immense task, it would be interesting to examine the speeds of convergence, as our inspiring paper did somewhat. There were a number of times throughout this work where we made note of interesting convergence speeds. Such as when two initial guess values were the same distance from their respective roots and yet they converged at different speeds (determined the root within a different number of iterations). This is intriguing and it would be amusing to explore if there's an explanation for this or possibly a pattern to identify. There may not be significant consequence of exploring the apparent discrepancies in convergence speed but, as mentioned any work in the subject of complex-valued functions and mathematical chaos is consequential.

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